TWO-PARAMETER MOTIONS IN MAGNETO-GASDYNAMICS (GROMEKA AND CHAPLYGIN TRANSFORMATIONS)

(DVUKHPARAMETRICHESKIE DVIZHENIIA V MAGNITNOI GAZODINAMIKE)

(PREOBRAZOVANIIA GROMEKI I CHAPLYGINA)

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The system of equations of magneto-gasdynamics in the steady two-parameter case with the electric field in a certain given direction reduces to a system of two scalar equations for two scalar unknown functions, after the Gromeka transformation is applied.

For rotational motion (under some additional restrictions), the system obtained reduces to a linear system, after the Chaplygin transformation is used. The basic physical properties of such flows are studied, and conditions on their ellipticity are obtained. Several limiting cases are considered.

1. Symmetry integrals of magneto-gasdynamics. The system of equations of ideal magneto-gasdynamics for adiabatic motion may be written in the form [1]

$$\begin{gathered} \partial \mathbf{H} / \partial t = \operatorname{rot} \mathbf{V} \times \mathbf{H}, \qquad \operatorname{div} \mathbf{H} = 0 \\ \partial \rho / \partial t + \operatorname{div} \rho \mathbf{V} = 0, \quad \partial \rho s / \partial t + \operatorname{div} \rho s \mathbf{V} = 0 \\ \partial \mathbf{V} / \partial t + \nabla (\mathbf{V}^2 / 2 + F) + (1 / \rho) \nabla p = \mathbf{V} \times \operatorname{rot} \mathbf{V} - (1 / 4\pi\rho) \mathbf{H} \times \operatorname{rot} \mathbf{H} \end{gathered} \tag{1.1}$$

We shall consider the pressure p to be an arbitrary function of the density ρ and the entropy s. In the steady case $(\partial/\partial t = 0)$, after integrating the "freezing of magnetic lines" equation, we may re-write the system (1.1) as

$$\begin{gathered} \operatorname{div} \mathbf{H} = 0, \qquad \operatorname{div} \rho \mathbf{V} = 0, \qquad \operatorname{div} \rho s \mathbf{V} = 0, \qquad \mathbf{V} \times \mathbf{H} = c \nabla \Phi \\ \nabla (\mathbf{V}^2 / 2 + F) + (1 / \rho) \nabla p = \mathbf{V} \times \operatorname{rot} \mathbf{V} - (1 / 4 \pi\rho) \mathbf{H} \times \operatorname{rot} \mathbf{H} \end{gathered} \tag{1.2}$$

where Φ is the electric field potential ($\mathbf{E} = -\nabla\Phi$).

Let (q_1, q_2, q_3) be a Cartesian coordinate system, and let all the basic physical quantities: velocity, magnetic field, density, entropy, and also potentials Φ and F, be independent of the third coordinate. Following Gromeka [2-6], the two-parameter solenoidal fields $\mathbf{H}$ and $\rho\mathbf{V}$ are represented by

$$\mathbf{H} = \nabla\psi \times \mathbf{e} + h\mathbf{e}, \qquad \mathbf{V} = (1/\rho)\nabla\psi_0 \times \mathbf{e} + v\mathbf{e} \tag{1.3}$$

Here the generalized stream functions ψ and ψ_0, as well as the third components of the velocity and magnetic field v and h, are arbitrary functions of the coordinates (q_1, q_2); $\mathbf{e}$ is the unit vector in the third direction.

Substituting (1.3) in the (adiabatic) energy equation and in the third component of the "frozen magnetic lines" equation (1.2), and solving the obtained Jacobian equation for the case when a transverse velocity component is present ($\psi_0' \neq 0$), we get

$$\psi = \psi(\xi), \qquad \psi_0 = \psi_0(\xi), \qquad s = s(\xi), \qquad \xi = \xi(q_1, q_2) \tag{1.4}$$

Here ψ, ψ_0, s and ξ are arbitrary functions of their arguments.

Substituting (1.4) into the first and second components of the frozen field equation, and into the third component of the momentum equation, in (1.2), we obtain, after integration

$$\rho\psi' v - \psi_0' h = c\rho\Phi', \qquad 4\pi\psi_0' v - \psi' h = 4\pi Q'$$

Everywhere, the prime (′) denotes differentiation with respect to the argument. The electric potentials $\Phi = \Phi(\xi)$ and $Q = Q(\xi)$ are arbitrary functions of the variable ξ. If the determinant of the above linear system in v and h does not equal zero: $4\pi(\psi_0')^2 \neq \rho(\psi')^2$, then solving, we find v and h

$$v = \frac{4\pi\psi_0' Q' - c\rho\psi'\Phi'}{4\pi(\psi_0')^2 - \rho(\psi')^2}, \qquad h = 4\pi\rho\frac{\psi' Q' - c\psi_0'\Phi'}{4\pi(\psi_0')^2 - \rho(\psi')^2} \tag{1.5}$$

Using relations (1.3) and (1.4), and the fact that for any two-parameter solenoidal field $\mathbf{H}$ the relation $\operatorname{rot}\mathbf{H} = \nabla h \times \mathbf{e} - \Delta\psi\mathbf{e}$ obtains, we rewrite the first two components of the momentum equation in (1.2) into the form

$$\begin{aligned}\nabla\left[\frac{1}{2}\left(\frac{\nabla\psi_0}{\rho}\right)^2 + F\right] + \frac{1}{\rho}\nabla\left(p + \frac{h^2}{8\pi}\right) = \\ = \left\{\psi_0'\left[\frac{1}{\rho^2}\Delta\psi_0 + \frac{1}{\rho}\left(\nabla\psi_0\nabla\frac{1}{\rho}\right)\right] - \frac{1}{4\pi\rho}\psi'\Delta\psi\right\}\nabla\xi\end{aligned} \tag{1.6}$$

We introduce the "effective transverse" pressure P, equal to the sum of the longitudinal magnetic pressure (that of the magnetic field along the third axis) and the gas pressure

$$P(\rho, \xi) = p(\rho, s) + h^2/8\pi \tag{1.7}$$

Then, in the general case where the density and entropy are functionally independent, $\partial(\rho, \xi)/\partial(q_1, q_2) \neq 0$, the general solution of (1.6) has the form

$$\begin{gathered}\psi_0'\left[\frac{1}{\rho^2}\Delta\psi_0 + \frac{1}{\rho}\left(\nabla\psi_0\nabla\frac{1}{\rho}\right)\right] - \frac{1}{4\pi\rho}\psi'\Delta\psi = \frac{\partial}{\partial\xi}\left(\frac{P}{\rho} + w - \int^{\rho}\frac{\partial P}{\partial\rho}\frac{d\rho}{\rho}\right) \\ \frac{1}{2}\left(\frac{\nabla\psi_0}{\rho}\right)^2 + F = w(\xi) - \int^{\rho}\frac{\partial P}{\partial\rho}\frac{d\rho}{\rho}\end{gathered} \tag{1.8}$$

Here w is an arbitrary function of ξ; hence, the lower limit of integration may be omitted. The relation (1.8) represents an exact solution of the vector equation (1.6); substitution of (1.8) in (1.6) results in an identity.

Physically, the first relation in (1.8) represents the law of change of the third component of the vorticity. The second relation in (1.8), representing the law of change of energy per unit mass, is the generalized Bernoulli integral for the case of rotational motion of a conducting gas.

Thus, if one of the Cartesian coordinates is ignorable, then in the steady case, the problem reduces (under very general hypotheses) to the solution of a system of two scalar differential equations (1.8) for the determination of ρ and ξ.

We remark that certain methods, similar to those used above, have been used by other authors [8-14].

2. Chaplygin transformation. Let there be no external body force ($F = 0$). We cross multiply the equations in (1.8), and multiply them by an arbitrary function of ρ and ξ. In order that the resulting equation may be reduced to "canonical" form

$$\frac{\partial}{\partial q_1}\left[\mathscr{E}(\rho, \xi)\frac{\partial\xi}{\partial q_1}\right] + \frac{\partial}{\partial q_2}\left[\mathscr{E}(\rho, \xi)\frac{\partial\xi}{\partial q_2}\right] = 0 \tag{2.1}$$

it suffices that the following condition hold:

$$\frac{(\psi_0')^2}{\rho}\frac{\partial}{\partial\xi}\left[\frac{P}{\rho} + w - \int^{\rho}\frac{\partial P}{\partial\rho}\frac{d\rho}{\rho}\right] + \left[w - \int^{\rho}\frac{\partial P}{\partial\rho}\frac{d\rho}{\rho}\right]\frac{\partial}{\partial\xi}\left[\frac{(\psi_0')^2}{\rho} - \frac{(\psi')^2}{4\pi}\right] = 0. \tag{2.2}$$

Condition (2.2) is found by identifying the equation obtained from (1.8) with equation (2.1) and eliminating $\mathscr{E}(\rho, \xi)$. The general solution of (2.2) has the form

$$P = \rho^2 \frac{\partial}{\partial\rho}\frac{\Pi}{\rho}, \qquad w = 0, \qquad \Pi \equiv \Pi(\mathscr{E}), \qquad \mathscr{E} \equiv \frac{\psi_0'^2}{\rho} - \frac{\psi'^2}{4\pi} \tag{2.3}$$

where Π is an arbitrary function of $\mathscr{E}$. Now the parameter ξ does not come out to be physically arbitrary: to each form of the functions $\psi = \psi_0(\xi)$ and $\psi = \psi(\xi)$ corresponds a particular character of the fields of the physical variables. The relation (2.3) imposes a restriction on the form of the equation of state $p = p(\rho, s)$ and on the nature of the motion. In what follows, we shall consider only those motions and states of the gas which satisfy condition (2.3).

Solving the canonical equations (2.1), and also using the second equation of (1.8) and the condition (2.3), we obtain the "canonical" system

$$\nabla\varphi = \mathscr{E}\nabla\xi \times \mathbf{e}, \qquad (\nabla\varphi)^2 = 2(\mathscr{E}u)^2, \qquad u^2 \equiv \Pi'(\mathscr{E}) \tag{2.4}$$

to which the Chaplygin transformation may be applied [15]. Apparently, the first to apply this transformation to magneto-gasdynamics was Nochevkina [16], and somewhat later, it was also used by several other authors [17-19]. Applying to system (2.4) a contact transformation [7, 15], we get

$$\frac{\partial\xi}{\partial\theta} + \frac{u}{\mathscr{E}}\frac{\partial\varphi}{\partial u} = 0, \qquad \frac{\partial\xi}{\partial u} + u\left(\frac{d}{du}\frac{1}{u\mathscr{E}}\right)\frac{\partial\varphi}{\partial\theta} = 0 \tag{2.5}$$

where θ is the angle between the vector $\nabla\phi$ and the q_1-axis. The connection with the space variables is given by the relation

$$d(q_1 + iq_2) = \frac{e^{i\theta}}{u}\left(id\xi + \frac{1}{\mathscr{E}}d\varphi\right) \tag{2.6}$$

The present transformation is mathematically equivalent to a transformation into "the hodograph plane", although the motions considered need not be plane motions (the velocity has a third component which, in general, is non-zero, $v \neq 0$).

Thus, (2.5) represents the desired system of linear homogeneous equations (for the functions ϕ and ξ), the coefficients of which depend on the independent variable $\mathscr{E}$ but not on θ. The condition of ellipticity is:

$$d(u\mathscr{E})^2 / du^2 > 0 \tag{2.7}$$

To analyze and solve this system in various limiting cases, well-known methods [7,15] may be successfully employed. In the general case, it appears natural that the results of Ovsiannikov [20,21] may be employed.

The cases considered in this section (and also later) are natural generalizations of some earlier works of similar nature. These include, first of all, the results for plane rotational flow of ordinary gas obtained by Sedov [7] and Rudnev [22,23], and also the later studies of Nochevkina [16] and Iur'ev [17] for various limiting cases of magnetogasdynamics.

3. Some simplest physical properties of "Chaplygin" flows. Using relations (1.5) and (2.3), we reduce the expression (1.3) for the velocity and magnetic field to the form

$$\begin{aligned} \rho\mathscr{E}\mathbf{V} &= \psi_0'\nabla\varphi + (\psi_0' Q' - c\rho\psi'\Phi')\,\mathbf{e} \\ \mathscr{E}\mathbf{H} &= \psi'\nabla\varphi + (\psi' Q' - c\psi_0'\Phi')\,\mathbf{e} \end{aligned} \tag{3.1}$$

We introduce the "transverse" magnetic energy (μ_H) and kinetic energy (μ_V) per unit volume

$$\mu_H \equiv (H_1^2 + H_2^2)/8\pi, \qquad \mu_V \equiv \rho(V_1^2 + V_2^2)/2 \tag{3.2}$$

Then, substituting (3.1) in (3.2) and using (2.4), we get

$$\mu_H = \frac{(\psi')^2}{4\pi}\frac{d\Pi}{d\mathscr{E}}, \qquad \mu_V = \frac{(\psi_0')^2}{\rho}\frac{d\Pi}{d\mathscr{E}} \tag{3.3}$$

We consider the difference (μ) and ratio (ϵ) of the "transverse" energies per unit volume

$$\mu \equiv \mu_V - \mu_H, \qquad \varepsilon \equiv \mu_V/\mu_H \tag{3.4}$$

Substituting (3.3) into (3.4), we obtain

$$\mu = \mathscr{E}\frac{d\Pi}{d\mathscr{E}}, \qquad \varepsilon = \frac{4\pi}{\rho}\left(\frac{d\psi_0}{d\psi}\right)^2 \tag{3.5}$$

Therefore, the difference of the energies is shown to depend only on the basic parameter ($\mathscr{E}$) of the theory, $\mu = \mu(\mathscr{E})$. Moreover, $\mathscr{E}$ is positive, if the density of the transverse kinetic energy exceeds the density of the transverse magnetic energy; $\mathscr{E}$ is negative if the density of the kinetic energy is less than that of the magnetic energy. We recall that all considerations are based on $\mathscr{E} \neq 0$. From relations (2.3) and (3.3), we find

$$\Pi = -(\mu_V + P) \tag{3.6}$$

Substituting (1.7) into (3.6), we get

$$\Pi = -(\mu_V + p + h^2/8\pi) \tag{3.7}$$

i.e. the quantity Π is taken with opposite sign to the sum of the transverse kinetic energy, the thermal energy, and the longitudinal magnetic energy, per unit volume.

Substituting (2.4) into (2.5), we reduce the Chaplygin equation to the form

$$\frac{\partial\xi}{\partial\theta} + \frac{2\Pi'}{\mathscr{E}\Pi''}\frac{\partial\varphi}{\partial\mathscr{E}} = 0, \qquad \frac{\partial\xi}{\partial\mathscr{E}} = \frac{1}{\mathscr{E}^2}\left(1 + \frac{\mathscr{E}\Pi''}{2\Pi'}\right)\frac{\partial\varphi}{\partial\theta}\,. \tag{3.8}$$

Substituting (2.4) into (2.7), the ellipticity condition for system (3.8) may be written as

$$-2\Pi\ /\mathscr{E}\Pi'' < 1 \tag{3.9}$$

In what follows, we shall call "constrained" those parts of the flow in which $d\Pi/d\mathscr{E} = 0$. Consequently, at the "constraint points", the transverse components of the velocity and magnetic field vanish. The roots of the equation $\Pi'(\mathscr{E}) = 0$ will be denoted by $\mathscr{E}_*$. The physical quantities corresponding to $\mathscr{E} = \mathscr{E}_*$ will be denoted by a lower asterisk and will be called the parameters of the constrained flow. We observe that the conventional [7,15] "constrained" flows are contained as special cases in those given here.

4. Motion in longitudinal magnetic field. Let there be no transverse component of the magnetic field, $\psi' = 0$. Then we see from (2.3)

$$\mathscr{E} \equiv (\psi_0')^2/\rho \tag{4.1}$$

From (3.3), (3.5), and (4.1), it follows that the transverse kinetic energy per unit volume is

$$\mu_V = \mathscr{E}\frac{d\Pi}{d\mathscr{E}} \tag{4.2}$$

In addition, from (3.5), (3.6), and (4.2), we get

$$P = -\frac{d}{d\mathscr{E}}(\mathscr{E}\Pi) \tag{4.3}$$

From (4.2) and (4.3), we conclude that both the effective pressure and the transverse kinetic energy depend only on the basic parameter $\mathscr{E}$

If the pressure is a function of the density to some power, $p \approx \rho^\gamma$, then comparing the expressions for the effective pressure, from (1.5), (1.7), (4.1) and (4.3), after integrating and identifying, we obtain a relation which is general for $\gamma \neq 2$:

$$p = p_*\left(\frac{\mathscr{E}_*}{\mathscr{E}}\right)^{\gamma}, \quad h = h_*\frac{\mathscr{E}_*}{\mathscr{E}}, \quad \Pi(\mathscr{E}) = \frac{p_*}{\gamma-1}\left[\left(\frac{\mathscr{E}_*}{\mathscr{E}}\right)^{\gamma} - \gamma\frac{\mathscr{E}_*}{\mathscr{E}}\right] + \frac{h_*^2}{8\pi}\left[\left(\frac{\mathscr{E}_*}{\mathscr{E}}\right)^2 - 2\frac{\mathscr{E}_*}{\mathscr{E}}\right] \tag{4.4}$$

The constants h_*, p_* and $\mathscr{E}_*$ are the parameters of the constrained flow. From (4.2) and (4.4), it follows that real pressures are described by sufficiently large values of the basic parameter $\mathscr{E}_* \leqslant \mathscr{E}$. Substituting (4.4) into (3.9), the ellipticity condition is reduced to the form

$$\frac{\gamma p_*}{\gamma-1}\left[\frac{\gamma+1}{2}\left(\frac{\mathscr{E}_*}{\mathscr{E}}\right)^{\gamma-1} - 1\right] + \frac{h_*^2}{4\pi}\left(\frac{3\mathscr{E}_*}{2\mathscr{E}} - 1\right) > 0 \tag{4.5}$$

If the longitudinal magnetic field is sufficiently small ($h_*^2/8\pi \ll p_*$), then from (4.5) follows the approximate condition of ellipticity, similar to the well-known expression in ordinary gas dynamics

$$\left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} < \frac{\mathscr{E}_*}{\mathscr{E}} \leqslant 1 \tag{4.6}$$

In the case of a strong longitudinal magnetic field ($p_* \ll h_*^2/8\pi$), the following approximate ellipticity condition follows from (4.5):

$$2/3 < \mathscr{E}_*/\mathscr{E} \leqslant 1 \tag{4.7}$$

Substituting (4.1) and (4.4) into (3.1), we obtain the following expressions for the velocity and magnetic field:

$$\mathbf{V} = \frac{d\xi}{d\psi_0}\nabla\varphi + v(\xi)\,\mathbf{e}, \qquad \mathbf{H} = h_*\frac{\mathscr{E}_*}{\mathscr{E}}\mathbf{e} \tag{4.8}$$

where $v = v(\xi)$ is an arbitrary function of ξ.

If the pressure is proportional to the square of the density ($\gamma = 2$), then from analyzing relations (1.5), (1.7), (4.1) and (4.3), after integrating and using some transformations, we get:

$$h = h_0\frac{\mathscr{E}_*}{\mathscr{E}}, \quad p = \left(P_* - \frac{h_0^2}{8\pi}\right)\left(\frac{\mathscr{E}_*}{\mathscr{E}}\right)^2, \quad \Pi = \frac{\mathscr{E}_*}{\mathscr{E}}\left(\frac{\mathscr{E}_*}{\mathscr{E}} - 2\right)P_* \tag{4.9}$$

The constants P_* and $\mathscr{E}_*$ are parameters of constrained flow, and $h_0 \equiv h_0(\xi)$ is an arbitrary function of ξ. Substituting (4.9) into (3.9), we arrive at the ellipticity condition (4.6), (4.7), which coincide for $\gamma = 2$.

Thus, a longitudinal magnetic field does not introduce any qualitative difference to the gasdynamical condition of ellipticity of the flow.

We remark that many results, concerning the rotational flow of a conducting gas in a magnetic field perpendicular to the plane of the flow, are contained in the work of Nochevkina [16].

Setting the magnetic field and the third component of the velocity to zero, we arrive at the relations equivalent to the corresponding results of Sedov [7] and Rudnev [22,23].

5. Ordinary gas dynamics. Setting $h_* = 0$ in the relations (4.1) to (4.6) and (4.8), we obtain the corresponding results for the adiabatic motion of a non-conducting gas. Comparing the well-known [7] equation of state of an ideal, perfect, gas

$$p = \mathrm{const}\left(\rho \exp \frac{S}{c_p}\right)^{\gamma}, \qquad \gamma = \frac{c_p}{c_v} \tag{5.1}$$

with (4.1) and (4.4), we see that, without loss of generality, the parameters $\mathscr{E}$ and ξ may be chosen in the following manner:

$$\mathscr{E} = \frac{1}{\rho} \exp\left(-\frac{S}{c_p}\right), \qquad \frac{d\psi_0}{d\xi} = \exp\left(-\frac{S}{2c_p}\right) \tag{5.2}$$

Substituting (5.2) into (4.8), we obtain an expression for the velocity:

$$\mathbf{V} = \exp\left(\frac{S}{2c_p}\right)\nabla\varphi + v(\xi)\,\mathbf{e} \tag{5.3}$$

We obtain the condition of ellipticity for the flow, upon substituting (5.2) into (4.5)

$$\left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} < \mathscr{E}_* \rho \exp \frac{S}{c_p} \leqslant 1 \tag{5.4}$$

where $\mathscr{E}_*$ is the parameter of the constrained flow. The relation (5.4) represents a generalization of a similar [7] condition for isentropic flow. In the case of plane problems, Sedov [7] and Rudnev [22,23] have arrived at similar results by somewhat different means.

6. Motion with a magnetic field of arbitrary orientation. Let the pressure be a function of the density to some power, $p \approx \rho^{\gamma}$. Then by analyzing the relations (1.5), (1.7) and (2.3), it follows that for arbitrary γ in the general case $\psi' = \mathrm{const}$. If there is a transverse component of the magnetic field ($\psi' \neq 0$), then, without loss of generality, we may set

$$d\xi = d\psi / \sqrt{4\pi} \tag{6.1}$$

Then, from (2.3), (3.5) and (6.1), we obtain

$$\mathscr{E} = \varepsilon - 1 \tag{6.2}$$

From (2.3) and (6.2), we obtain the following exprḙssion for the effective pressure:

$$P = -\frac{d}{d\varepsilon}(\varepsilon\Pi) \tag{6.3}$$

The longitudinal magnetic pressure and the parameter Π are given by the expressions

$$p = p_*\left(\frac{\varepsilon_*}{\varepsilon}\right)^\gamma, \qquad h = h_*\frac{\varepsilon_* - 1}{\varepsilon - 1} \tag{6.4}$$

$$\Pi = \frac{p_*}{\gamma - 1}\left[\left(\frac{\varepsilon_*}{\varepsilon}\right)^\gamma - \gamma\frac{\varepsilon_*}{\varepsilon}\right] + \frac{h_*^2}{8\pi}\left[\frac{(\varepsilon_* - 1)^2}{\varepsilon - 1} - \frac{\varepsilon_*^2}{\varepsilon}\right]$$

The constants h_*, p_* and ϵ_* are parameters of constrained flow. From the relations (3.1), (6.1), (6.2) and (6.4), we find the expressions for the magnetic field and velocity

$$\begin{gathered}(\varepsilon - 1)\,\mathbf{H} = \sqrt{4\pi}\,\nabla\varphi + h_*(\varepsilon_* - 1)\,\mathbf{e}\\ (\varepsilon - 1)\,\mathbf{V} = \frac{\varepsilon}{\sqrt{4\pi}}\frac{d\psi}{d\psi_0}\nabla\varphi + (\varepsilon - 1)\,v(\xi, \varepsilon)\,\mathbf{e}\end{gathered} \tag{6.5}$$

From the analysis of relations (3.8), (3.9) and (6.4), we obtain a hypersurface (in the α, β, γ, ϵ space), separating the region of elliptic flows from the region of hyperbolic flows

$$\gamma\,[(\gamma - 1)\,\varepsilon - (\gamma + 1)] + 2\beta\varepsilon^{\gamma - 1} = 0, \qquad \frac{2\beta}{\varepsilon^3} = \frac{\gamma(\gamma + 1)}{\varepsilon^{\gamma + 2}} + \frac{2\alpha}{(\varepsilon - 1)^3} \tag{6.6}$$

$$\alpha \equiv (\gamma - 1)\frac{(\varepsilon_* - 1)^2 h_*^2}{\varepsilon_*^\gamma\, 8\pi p_*}, \qquad \beta \equiv (\gamma - 1)\,\varepsilon_*^{2-\gamma}\left(\frac{\gamma}{\gamma - 1}\varepsilon_*^{-1} + \frac{h_*^2}{4\pi p_*}\right)$$

We introduce the notations

$$a \equiv 1/\varepsilon, \qquad a_* \equiv 1/\varepsilon_*, \qquad A_* \equiv h_*^2/8\pi p_* \tag{6.7}$$

Then for a monotomic gas ($\gamma = 5/3$), the hypersurface (6.6) in the (a, a_*, A_*) space assumes the form

$$A_* = \frac{5}{6}\left(\frac{a_*}{a}\right)^{1/3}(4a - 1) - \frac{5}{2}a_*, \qquad A_* = \frac{5a_*^{1/3}}{6}\frac{4a^{2/3} - 3a_*^{2/3}}{1 - (1 - a_*)^2/(1 - a)^3}$$

From an analysis of (6.6), it follows that there exist several zones of elliptic and hyperbolic flows, mutually alternating. In particular, for velocities of ordered motion smaller than that of thermal motion, hyperbolic flow is possible; and for those larger, elliptic flow is possible. Such conclusions (under some additional restrictions) are contained in the investigations of Iur'ev [17] and Kogan [24].

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